CLEARINGHOUSE FOR FEDERAL SCIENTIFIC AND TECHNICAL INFORMATION CFSTI
DOCUMENT MANAGEMENT BRANCH 410.11

## LIMITATIONS IN REPRODUCTION QUALITY

ACCESSION # AD 666 613

- [x] 1. WE REGRET THAT LEGIBILITY OF THIS DOCUMENT IS IN PART UNSATISFACTORY. REPRODUCTION HAS BEEN MADE FROM BEST AVAILABLE COPY.
- [ ] 2. A PORTION OF THE ORIGINAL DOCUMENT CONTAINS FINE DETAIL WHICH MAY MAKE READING OF PHOTOCOPY DIFFICULT.
- [ ] 3. THE ORIGINAL DOCUMENT CONTAINS COLOR, BUT DISTRIBUTION COPIES ARE AVAILABLE IN BLACK-AND-WHITE REPRODUCTION ONLY.
- [ ] 4. THE INITIAL DISTRIBUTION COPIES CONTAIN COLOR WHICH WILL BE SHOWN IN BLACK-AND-WHITE WHEN IT IS NECESSARY TO REPRINT.
- [ ] 5. LIMITED SUPPLY ON HAND: WHEN EXHAUSTED, DOCUMENT WILL BE AVAILABLE IN MICROFICHE ONLY.
- [ ] 6. LIMITED SUPPLY ON HAND: WHEN EXHAUSTED DOCUMENT WILL NOT BE AVAILABLE.
- [ ] 7. DOCUMENT IS AVAILABLE IN MICROFICHE ONLY.
- [ ] 8. DOCUMENT AVAILABLE ON LOAN FROM CFSTI ( TT DOCUMENTS ONLY).
- [ ] 9.

NBS 9/64

PROCESSOR: [illegible]

# NOTE ON THE MINIMUM VARIANCE OF UNBIASED ESTIMATES OF DOPPLER SHIFT

P. Swerling

P-1247

January 7, 1958



The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

## SUMMARY

The methods developed in another paper (Ref. 1) are applied to the calculation of the minimum variance of unbiased estimates of doppler shift, when the received waveform is observed against a background of additive white Gaussian noise.

The purpose of this note is to illustrate the application of the methods developed in Refs. 1 and 2 to the calculation of the minimum variance of unbiased estimates of doppler shift in a received wave form. It would be possible to evaluate the function G of Ref. 1, and hence to evaluate the greatest lower bound for the variance of estimates of doppler shift, for the general case where the received amplitude and time delay, as well as the doppler shift, are unknown. This would not result in a simple expression for the answer, however; therefore, it will be assumed here that the received energy and the time delay are known. To be precise, we assume the observed wave-form to be

$$v(t) = \alpha_0 e^{\zeta/2} F(e^{\zeta} t) + n(t) \qquad (1)$$

for $0 \leq t < \infty$ . Here $\alpha_0$ is assumed to be known; time delay has been set equal to zero (no loss of generality for cases where time delay is assumed known), and $n(t)$ is white Gaussian noise with spectral density $N_0$.

The parameter $\zeta$ is a measure of the 'doppler shift'. For example, if we are dealing with a reflected radar signal from a target moving with radial velocity $v$ , and if we assume $\frac{v}{c}$ to be small ($c$ = velocity of light) then approximately $\zeta \approx \frac{2v}{c}$ .

It will henceforth be assumed that the a priori range of variation of $\zeta$ is a real interval $[a, b]$ , and that $|a| \ll 1$, $|b| \ll 1$.

The factor $e^{\zeta/2}$ multiplying $F$ makes the total received energy independent of the value of $\zeta$ . This would not actually be the case, since the received energy actually would be greater for approaching targets and less for receding targets. However, the factor $e^{\zeta/2}$ makes for a great simplification in the mathematics, so it will be assumed that the

received energy is in fact independent of $\zeta$. In cases of practical interest, this assumption probably has little effect on the calculated value of the minimum variance of unbiased estimates of $\zeta$.

The evaluation of $G(\zeta, \zeta' \mid \zeta_0)$ can be carried out in a manner virtually identical with that of Ref. 1, the result being as follows:

Let

$$R = \frac{2\alpha_0^2}{N_0} \int_0^\infty F^2(t)\, dt \tag{2}$$

$$\phi(\zeta) = \frac{e^{\zeta/2} \int_0^\infty F(t)\, F(e^{\zeta} t)\, dt}{\int_0^\infty F^2(t)\, dt} \tag{3}$$

$$H(\zeta) = \exp\left[R\, \phi(\zeta)\right] \tag{4}$$

(where exp stands for the exponential function); then

$$G(\zeta, \zeta' \mid \zeta_0) = \frac{e^R\, H(\zeta - \zeta')}{H(\zeta - \zeta_0)\, H(\zeta' - \zeta_0)} \tag{5}$$

It is seen that this is of precisely the same form as Eq. 20 of Ref. 1, with $\alpha = \alpha' = \alpha_0$, except that the definition of $H$ is given by Eqs. (2), (3), and (4) of the present note. Therefore, all the developments of Ref. 1 can be followed exactly, finally resulting in:

Let

$$L(\zeta) = H(\zeta) - 1 \tag{6}$$

$$\mathcal{L}(u) = \int_{-\infty}^{\infty} e^{-iu\zeta} L(\zeta)\, d\zeta \tag{7}$$

then $\sigma^2_{glb}(\zeta_0)$, the greatest lower bound for the variance of unbiased estimates of $\zeta$ when the true value is $\zeta_0$, is given approximately by Eq. 45 of Ref. 1 with $\zeta_0$ substituted for $\tau_0$, and of course with the new definitions of $\mathcal{L}(u)$, $L(\zeta)$, R, etc.

In particular for sufficiently large R,

$$\sigma^2_{glb}(\zeta_0) \approx \frac{e^{-R}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\, du \tag{8}$$

We will now turn to the evaluation of $\phi(\zeta)$ in a large class of cases. Namely, we assume F of the form

$$F(t) = F_1(t) \cos \omega_0 t + F_2(t) \sin \omega_0 t \tag{9}$$

It can then be established with some elementary trigonometric identities that

$$\phi(\zeta) = \frac{e^{\zeta/2}}{\int_0^{\infty} F^2(t)\, dt} \int_0^{\infty} \mathcal{F}(t)\, dt \tag{10}$$

where

$$\mathcal{F}(t) = \frac{1}{2} F_1(t)\, F_1(e^{\zeta} t) \left\{ \cos\left[\omega_o(e^{\zeta} -1)t\right] + \cos\left[\omega_o(e^{\zeta} +1)t\right] \right\}$$
$$+ \frac{1}{2} F_2(t)\, F_2(e^{\zeta} t) \left\{ \cos\left[\omega_o(e^{\zeta} -1)t\right] - \cos\left[\omega_o(e^{\zeta} +1)t\right] \right\}$$
$$+ \frac{1}{2} F_1(t)\, F_2(e^{\zeta} t) \left\{ \sin\left[\omega_o(e^{\zeta} -1)t\right] + \sin\left[\omega_o(e^{\zeta} +1)t\right] \right\}$$
$$- \frac{1}{2} F_2(t)\, F_1(e^{\zeta} t) \left\{ \sin\left[\omega_o(e^{\zeta} -1)t\right] - \sin\left[\omega_o(e^{\zeta} +1)t\right] \right\} \qquad (11)$$

Thus, $\phi(\zeta)$ can be readily evaluated for cases in which the Fourier cosine transforms of $F_1(t)\, F_1(e^{\zeta} t)$ and $F_2(t)\, F_2(e^{\zeta} t)$, and the Fourier sine transforms of $F_1(t)\, F_2(e^{\zeta} t)$ and $F_2(t)\, F_1(e^{\zeta} t)$ are known.

Also, in most cases, if $\omega_o$ is sufficiently large, the terms involving $\omega_o(e^{\zeta} +1)t$ can all be neglected.

As an illustration, suppose

$$F_1(t) = \exp\left[-\frac{1}{2}\beta^2 t^2\right] \qquad (0 \leq t < \infty)$$
$$F_2(t) = 0 \qquad (12)$$

Then, since $|\zeta|$ is always assumed $<< 1$, and assuming $\omega_o >> \beta$,

$$\phi(\zeta) \approx \exp\left[\frac{-\omega_o^2\, \zeta^2}{4\beta^2}\right] \qquad (13)$$

1)

Following Ref. 2, we can then readily evaluate $\sigma^2_{glb}$ for this case:

$$\sigma^2_{glb} \approx \frac{e^{-R}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\, du \approx \frac{2\beta^2}{R\,\omega_o^2} \tag{14}$$

It is also not difficult to obtain the result for

$F_1(t) = \exp\left[-\frac{1}{2}\beta^2 t^2\right]$, $F_2(t) = kt \exp\left[-\frac{1}{2}\gamma^2 t^2\right]$,

for example.

## REFERENCES

1. Swerling, P., 'A Method of Computing the Inherent Accuracy with Which a Time Delay Can be Estimated', The RAND Corporation, P-1185, September 27, 1957.

2. Swerling, P., 'Approximate Evaluation of an Expression Arising in the Theory of Time Delay Estimation', The RAND Corporation, P-1221, November 23, 1957.